

Neutrino Masses and Mixing with General Mass Matrices

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Abstract

We consider the most general neutrino masses and mixings including Dirac mass terms, M_D , as well as Majorana masses, M_R and M_L . Neither the Majorana nor the Dirac mass matrices are expected to be diagonal in the eigenbasis of weak interactions, and so the resulting eigenstates of the Hamiltonian are admixtures of $SU(2)_L$ singlet and doublet fields of different “generations.” We show that for three generations each of doublet and singlet neutrinos, diagonalization of the Hamiltonian to obtain the propagating eigenstates in the general case requires diagonalization of a 12×12 Hermitian matrix, rather than the traditional 6×6 symmetric mass matrix. The symmetries of the 12×12 matrix *are* such that it has 6 pairs of real eigenvalues. Although the standard “see-saw” mechanism remains valid, and indeed the eigenvalues obtained are identical to the standard ones, the correct description of diagonalization and mixing is more complicated. The analogs of the CKM matrix for the light and the heavy neutrinos are nonunitary, enriching the opportunities for CP violation in the full neutrino sector.

1 Introduction

One of the most intriguing features of the Standard Model is the complex mismatch between quark eigenstates of the weak interactions and the freely propagating eigenstates. Particularly intriguing is the resulting CP violation in the weak charged current interactions. In the Standard Model, this mismatch is confined to the quark sector, and by convention to the down-type quarks. The charged leptons, like the up-type quarks, can be chosen to have identical mass and weak eigenstates. Meanwhile, the absence of $SU(2)_L$ singlet neutrino fields precludes a Dirac mass for the neutrino, while a purely Majorana mass for the $SU(2)_L$ doublet neutrinos seems to be in conflict with experimental results on the width of the Z boson and the ρ -parameter. This eliminates the possibility of any neutrino masses or mixings, and any of the associated CP violation, unless new fields are added to the theory.

It has long been suspected that neutrinos are not absolutely massless. Many GUT theories and left-right symmetric theories imply the existence of $SU(2)_L \times U(1)_Y$ singlet fields, often called sterile neutrinos, and hence the possibility of non-zero masses for both

sterile and active ($SU(2)_L$ doublet) neutrinos. Over the past thirty years, considerable experimental evidence has accumulated to support the view that neutrinos are not massless. These include the large deficit in neutrinos arriving from the Sun compared to the predictions of the Standard Model of particle physics together with standard solar models. This deficit, it has been shown, cannot be explained by changes in the solar model. The only fully self-consistent explanation for all the solar neutrino data is that the electron neutrinos emitted in the Sun oscillate into other neutrino states with smaller cross-sections for detection. (For a recent review of the problem see [1] and references therein.) Such oscillations require non-zero neutrino masses and mixings. Similarly in the flux of neutrinos produced by cosmic rays in the atmosphere there is a deficit in the observed ratio of muon neutrinos to electron neutrinos compared to the predicted ratio [2], as well as an up-down asymmetry in the flux of muon neutrinos. These can be explained by the oscillation of muon neutrinos into tau or sterile neutrinos. The null hypothesis of no neutrino mixing is strongly rejected by both the Kamiokande and Super-Kamiokande data [2] which also are consistent with the preferred parameters for an explanation in terms of neutrino oscillations. One laboratory experiment, the LSND [3], also finds evidence for neutrino oscillations.

Given the overwhelming experimental evidence and theoretical motivation for neutrino masses and mixings, a full and careful treatment of the problem is warranted. Previous discussions have often focused, sometimes deliberately, on special cases, in particular on a real neutrino mass matrix. Although the masses obtained by the usual “see-saw” mechanism [5] remain unchanged in the full theory, the detailed mechanics of treating general mass models are more complicated. In particular, diagonalization of the Hamiltonian is not equivalent to diagonalization of the standard mass matrix. Also, the leptonic CKM matrix is *not* a 3×3 unitary matrix in generation space, but rather two 3×3 nonunitary matrices, one for the light and one for the heavy mass eigenstates (see section 6).

2 Mass eigenstates in neutrino theory

Suppose that the three $SU(2)_L$ doublet neutrinos of the Standard Model, ν_i ($i = e, \mu, \tau$), are supplemented by three $SU(2)_L$ singlet neutrinos N_i . ν_{ia} and N_{ia} are two-component Weyl fermions, with definite Lorentz properties – the index a signifying that they transform under the $(\frac{1}{2}, 0)$ representation. (In principle one can have more or less than three singlet neutrinos and this treatment generalizes in an obvious way.) The general free-field Lagrangian density for the neutrinos is

$$\begin{aligned}
-\mathcal{L} = & \quad iN_{ia}\sigma^{\mu\dot{a}b}\partial_\mu\bar{N}_{i\dot{b}} + i\bar{\nu}_{i\dot{a}}\bar{\sigma}^{\mu\dot{a}b}\partial_\mu\nu_{ib} + \bar{\nu}_{i\dot{a}}M_{Dij}\bar{N}_j^{\dot{a}} + N_i^a M_{Dij}^\dagger \nu_{ja} \\
& + \frac{1}{2}\bar{N}_{i\dot{a}}M_{Rij}\bar{N}_j^{\dot{a}} + \frac{1}{2}N_i^a M_{Rij}^\dagger N_{ja} + \frac{1}{2}\bar{\nu}_{i\dot{a}}M_{Lij}\bar{\nu}_j^{\dot{a}} + \frac{1}{2}\nu_i^a M_{Lij}^\dagger \nu_{ja}
\end{aligned} \tag{1}$$

Greek letters ($\mu = 0, 1, 2, 3$) denote Lorentz four-vector indices, with $\sigma_{ab}^\mu = (\mathbf{1}, \vec{\sigma})_{ab}$ and $\bar{\sigma}^{\mu\dot{a}b} = (\mathbf{1}, -\vec{\sigma})^{\dot{a}b}$. ($\vec{\sigma}$ are the usual Pauli matrices.) Latin letters denote either Weyl spinor indices ($a, b = 1, 2$) or generation indices ($i, j = 1, 2, 3$). The dotted spinor

indices (\dot{a}, \dot{b}) attached to $\bar{\nu}_i$ and \bar{N}_i indicate that the complex conjugates ν_i^* and N_i^* transform under the $(0, \frac{1}{2})$ representation of the Lorentz group — $(\nu_a)^* = \bar{\nu}_{\dot{a}}$ and $(N^a)^* = \bar{N}^{\dot{a}}$. Spinor indices, dotted and undotted, are raised and lowered by g_{ab} and $\bar{g}^{\dot{a}\dot{b}}$ which act as metric tensors in spinor space :

$$g_{ab} = i(\sigma^2)_{ab} = \varepsilon_{ab} \quad \text{and} \quad \bar{g}^{\dot{a}\dot{b}} = i(\bar{\sigma}^2)^{\dot{a}\dot{b}} = -\varepsilon^{\dot{a}\dot{b}} \quad (2)$$

with $\varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = \varepsilon^{12} = \varepsilon^{\dot{1}\dot{2}} = 1$. The inverses of g_{ab} and $\bar{g}^{\dot{a}\dot{b}}$ are respectively $g^{ab} = -\varepsilon^{ab}$ and $\bar{g}_{\dot{a}\dot{b}} = \varepsilon_{\dot{a}\dot{b}}$. Thus, $\sigma^{\mu\dot{a}b} = (\mathbf{1}, -\vec{\sigma}^*)^{\dot{a}b}$ and $(\sigma^{\mu\dot{a}b})^* = (\mathbf{1}, -\vec{\sigma})^{\dot{a}b} \equiv \bar{\sigma}^{\mu\dot{a}b}$

M_D , M_L and M_R are complex matrices. M_R and M_L are symmetric, since $\bar{N}_{i\dot{a}} M_{Rij} \bar{N}_j^{\dot{a}} = -\bar{N}_j^{\dot{a}} M_{Rij} \bar{N}_{i\dot{a}} = +\bar{N}_{j\dot{a}} M_{Rij} \bar{N}_i^{\dot{a}}$, and likewise for M_L .

It is convenient to make explicit in \mathcal{L} the dependence on spinor indices $a = 1, 2$. So long as no freely propagating eigenstates are massless, we can, without loss of generality, do this in the rest frame in which $\vec{\partial} N_{ia} = \vec{\partial} \bar{N}_i^{\dot{a}} = \vec{\partial} \nu_{ia} = \vec{\partial} \bar{\nu}_i^{\dot{a}} = 0$. This will allow us to find the zero-momentum eigenmodes of the system. Neutrinos of non-zero momentum are related to these by appropriate Lorentz boosts.

$$\begin{aligned} -\mathcal{L} = & iN_{i1}\partial_0 N_{i1}^* + iN_{i2}\partial_0 N_{i2}^* + i\nu_{i1}^*\partial_0 \nu_{i1} + i\nu_{i2}^*\partial_0 \nu_{i2} - \nu_{i1}^* M_{Dij} N_{j2}^* + \nu_{i2}^* M_{Dij} N_{j1}^* \\ & - N_{i2} M_{Dij}^\dagger \nu_{j1} + N_{i1} M_{Dij}^\dagger \nu_{j2} - N_{i1}^* M_{Rij} N_{j2}^* - N_{i2}^* M_{Rij}^\dagger N_{j1} - \nu_{i1}^* M_{Lij} \nu_{j2}^* - \nu_{i2}^* M_{Lij}^\dagger \nu_{j1} \end{aligned} \quad (3)$$

The equations of motion that follow from the Lagrangian density (3) are (suppressing generation indices):

$$\underbrace{\frac{i}{2} \frac{\partial}{\partial t} \begin{pmatrix} \nu_1 \\ N_1 \\ \nu_2^* \\ N_2^* \\ \nu_1^* \\ N_1^* \\ \nu_2 \\ N_2 \end{pmatrix}}_{\Psi} = \underbrace{\begin{pmatrix} 0 & 0 & M_L & M_D & 0 & 0 & 0 & 0 \\ 0 & 0 & M_D^T & M_R & 0 & 0 & 0 & 0 \\ M_L^\dagger & M_D^* & 0 & 0 & 0 & 0 & 0 & 0 \\ M_D^\dagger & M_R^\dagger & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -M_L^\dagger & -M_D^* \\ 0 & 0 & 0 & 0 & 0 & 0 & -M_D^\dagger & -M_R^\dagger \\ 0 & 0 & 0 & 0 & -M_L & -M_D & 0 & 0 \\ 0 & 0 & 0 & 0 & -M_D^T & -M_R & 0 & 0 \end{pmatrix}}_{H_0} \underbrace{\begin{pmatrix} \nu_1 \\ N_1 \\ \nu_2^* \\ N_2^* \\ \nu_1^* \\ N_1^* \\ \nu_2 \\ N_2 \end{pmatrix}}_{\Psi} \quad (4)$$

M_D , M_R and M_L should be interpreted as 3×3 matrices in generation space, while ν_a , ν_a^* , N_a and N_a^* ($a = 1, 2$) are each three-dimensional vectors in generation space. H_0 is the free, zero-momentum Hamiltonian operator of the theory. Its eigenstates are the zero-momentum normal modes — free-field propagating degrees of freedom in the theory. H_0 is also the “mass” matrix in the same basis, *i.e.*

$$\mathcal{L}_{mass} = \Psi^\dagger H_0 \Psi \quad (5)$$

This ensures that the zero-momentum free eigenmodes are in fact mass eigenstates, as would be expected. The eigenvalues are the masses of the physical neutrinos. However, it is crucial to note that H_0 is not the standard mass matrix, $\mathcal{M} = \begin{pmatrix} M_L & M_D \\ M_D^T & M_R \end{pmatrix}$.

H_0 is defined in both spinor and generation space, while \mathcal{M} acts only in generation space.

Fortunately, we are not required to manipulate the monolithic 24×24 H_0 , since

$$H_0 = \begin{pmatrix} H_0^{\text{red}} & 0 \\ 0 & -(H_0^{\text{red}})^* \end{pmatrix}. \quad (6)$$

The matrix H_0^{red} is Hermitian, and the complex conjugate of the unitary transformation which diagonalizes H_0^{red} will diagonalize $(H_0^{\text{red}})^*$. H_0^{red} acts on

$$\Psi^{\text{red}} \equiv \begin{pmatrix} \nu_1 \\ N_1 \\ \nu_2^* \\ N_2^* \end{pmatrix} \quad (7)$$

while H_0 acts on $\Psi \equiv (\Psi^{\text{red}} (\Psi^{\text{red}})^*)^T$. The complete diagonalization procedure will be presented in section (4).

3 The Case of One Generation

In the case of one generation the matrix which we must diagonalize is

$$\begin{pmatrix} 0 & 0 & m_L e^{i\theta} & m_D e^{i\varphi} \\ 0 & 0 & m_D e^{i\varphi} & m_R e^{i\phi} \\ m_L e^{-i\theta} & m_D e^{-i\varphi} & 0 & 0 \\ m_D e^{-i\varphi} & m_R e^{-i\phi} & 0 & 0 \end{pmatrix} \quad (8)$$

For the sake of simplicity, we take $m_L = 0$. The eigenvalues of this matrix are $(m_+, m_-, -m_-, -m_+)$ with

$$m_{\pm} = \frac{1}{2}(\sqrt{4m_D^2 + m_R^2} \pm m_R) \quad (9)$$

For $m_D \ll m_R$, $m_+ \simeq m_R$, while $m_- \simeq m_D^2/m_R$, the usual “see-saw” mechanism results [5]. The corresponding eigenvectors are:

$$\begin{pmatrix} \frac{e^{i(\varphi - \frac{\phi}{2})} m_-}{M_D} \\ e^{i\frac{\phi}{2}} \\ \frac{e^{i(\frac{\phi}{2} - \varphi)} m_-}{M_D} \\ e^{-i\frac{\phi}{2}} \end{pmatrix} \quad \begin{pmatrix} \frac{e^{i(\varphi - \frac{\phi}{2})} m_+}{M_D} \\ -e^{i\frac{\phi}{2}} \\ \frac{-e^{i(\frac{\phi}{2} - \varphi)} m_+}{M_D} \\ e^{-i\frac{\phi}{2}} \end{pmatrix} \quad \begin{pmatrix} \frac{-e^{i(\varphi - \frac{\phi}{2})} m_+}{M_D} \\ e^{i\frac{\phi}{2}} \\ \frac{-e^{i(\frac{\phi}{2} - \varphi)} m_+}{M_D} \\ e^{-i\frac{\phi}{2}} \end{pmatrix} \quad \begin{pmatrix} \frac{-e^{i(\varphi - \frac{\phi}{2})} m_-}{M_D} \\ -e^{i\frac{\phi}{2}} \\ \frac{e^{i(\frac{\phi}{2} - \varphi)} m_-}{M_D} \\ e^{-i\frac{\phi}{2}} \end{pmatrix} \quad (10)$$

Some interesting properties of these eigenvectors will be discussed in section (4) and (5).

4 The General Case

In general $M_D \neq 0$, $M_R \neq 0$, $M_L \neq 0$, and there are three generations of doublet neutrinos (although in many extensions of the standard model $M_L = 0$ identically). As discussed above, there can be any number of singlet neutrinos, but the Dirac mass terms will couple only three independent linear combinations of the singlet neutrinos to the doublet neutrinos. If there are fewer than three singlet neutrinos, then this is equivalent to appropriate zero entries in M_D , M_R and M_L . We will therefore assume that there are three singlet neutrino species. This leaves us to diagonalize a higher dimensional matrix which contains M_D , M_R and M_L . Let us, as is often done, introduce the new fields:

$$f_{\pm} \equiv \frac{\psi_L \pm (\psi_L)^c}{\sqrt{2}}, \quad \text{and} \quad F_{\pm} \equiv \frac{\psi_R \pm (\psi_R)^c}{\sqrt{2}} \quad (11)$$

and rewrite \mathcal{L} as:

$$\begin{aligned} -\mathcal{L} = & \frac{1}{2}i (\overline{f_+} \gamma^\mu \partial_\mu f_+ + \overline{F_+} \gamma^\mu \partial_\mu F_+ + \overline{f_-} \gamma^\mu \partial_\mu f_- + \overline{F_-} \gamma^\mu \partial_\mu F_-) + \quad (12) \\ & + \frac{1}{2} (\overline{f_+} \quad \overline{F_+} \quad \overline{f_-} \quad \overline{F_-}) \begin{pmatrix} M_L + M_L^\dagger & M_D + M_D^* & M_L - M_L^\dagger & M_D - M_D^* \\ M_D^T + M_D^\dagger & M_R + M_R^\dagger & M_D^\dagger - M_D^T & M_R - M_R^\dagger \\ M_L^\dagger - M_L & M_D - M_D^* & -M_L - M_L^\dagger & M_D + M_D^* \\ M_D^\dagger - M_D^T & M_R^\dagger - M_R & M_D^T + M_D^\dagger & -M_R - M_R^\dagger \end{pmatrix} \begin{pmatrix} f_+ \\ F_+ \\ f_- \\ F_- \end{pmatrix} \end{aligned}$$

M_D , M_R and M_L are 3×3 matrices in generation space. If all three of these matrices are real, then the cross terms between $+$ fields and $-$ fields vanish and the theory decouples into two sectors: “+” and “-”.

$$\begin{aligned} -\mathcal{L} = & \frac{1}{2}i \overline{f_+} \gamma^\mu \partial_\mu f_+ + \frac{1}{2}i \overline{F_+} \gamma^\mu \partial_\mu F_+ + \frac{1}{2} (\overline{f_+} \quad \overline{F_+}) \underbrace{\begin{pmatrix} M_L & M_D \\ M_D^T & M_R \end{pmatrix}}_{\mathcal{M}_+} \begin{pmatrix} f_+ \\ F_+ \end{pmatrix} + \quad (13) \\ & + \frac{1}{2}i \overline{f_-} \gamma^\mu \partial_\mu f_- + \frac{1}{2}i \overline{F_-} \gamma^\mu \partial_\mu F_- + \frac{1}{2} (\overline{f_-} \quad \overline{F_-}) \underbrace{\begin{pmatrix} -M_L & M_D \\ M_D^T & -M_R \end{pmatrix}}_{\mathcal{M}_-} \begin{pmatrix} f_- \\ F_- \end{pmatrix} \end{aligned}$$

The matrix $\mathcal{M}_+ \equiv \mathcal{M}$ is just a standard “see-saw” matrix. \mathcal{M}_- has eigenvalues with equal magnitudes but opposite signs from the eigenvalues of \mathcal{M}_+ . Making the change of variables $f_- \rightarrow -\gamma_5 f_-$ and $F_- \rightarrow \gamma_5 F_-$, which keeps kinetic terms invariant, we convert \mathcal{M}_- to \mathcal{M}_+ . The theory is thereby rewritten in terms of Majorana fields, $(f_{\pm})^c = f_{\pm}$ and $(F_{\pm})^c = F_{\pm}$, and the eigenstates of the mass matrices are Majorana fields as well.

We can not, or course, confine ourselves just to the “+” sector or just to the “-” sector, even though these two sectors have identical structures. Doing that we would lose half of original degrees of freedom. Also, eigenstates in these two sectors can have different properties (like helicity, P-parity, CP-parity, ...).

If the mass parameters are complex, then the cross terms in equation (12) do not vanish and the situation is more complicated. To find the eigenstates we must get rid of cross terms, *i.e.* diagonalize the theory.

We continue discussion in the basis where the connection to the two-component formalism is obvious.

$$\begin{aligned}
-\mathcal{L} = & i\bar{\psi}_L\gamma^\mu\partial_\mu\psi_L + i\bar{\psi}_R\gamma^\mu\partial_\mu\psi_R + \\
& + \frac{1}{2}(\bar{\psi}_L \ (\psi_R)^c) \begin{pmatrix} M_L & M_D \\ M_D^T & M_R \end{pmatrix} \begin{pmatrix} (\psi_L)^c \\ \psi_R \end{pmatrix} + \frac{1}{2}(\overline{(\psi_L)^c} \ \bar{\psi}_R) \begin{pmatrix} M_L^\dagger & M_D^* \\ M_D^\dagger & M_R^\dagger \end{pmatrix} \begin{pmatrix} \psi_L \\ (\psi_R)^c \end{pmatrix}
\end{aligned} \tag{14}$$

where we used $\bar{\psi}_L\psi_R = \overline{(\psi_R)^c}(\psi_L)^c$. In more compact notation \mathcal{L}_{mass} is:

$$-\mathcal{L}_{mass} = \frac{1}{2}(\overline{(n_R)^c} \ \bar{n}_R) \begin{pmatrix} 0 & \mathcal{M} \\ \mathcal{M}^\dagger & 0 \end{pmatrix} \begin{pmatrix} (n_R)^c \\ n_R \end{pmatrix} \tag{15}$$

where $n_R = ((\psi_L)^c \ \psi_R)^T$. The mass matrix in (15) is just H_0^{red} .

The first step in diagonalizing H_0^{red} is a unitary transformation with the matrix

$$S^{\text{red}} \equiv \begin{pmatrix} \mathcal{U}^T & 0 \\ 0 & \mathcal{U}^\dagger \end{pmatrix}, \tag{16}$$

where \mathcal{U} diagonalize \mathcal{M} : $\mathcal{U}^T \mathcal{M} \mathcal{U} = \mathcal{M}_d$ with \mathcal{M}_d a real, positive, diagonal matrix. So, we have:

$$S^{\text{red}} H_0^{\text{red}} (S^{\text{red}})^\dagger = \begin{pmatrix} 0 & \mathcal{M}_d \\ \mathcal{M}_d & 0 \end{pmatrix} \tag{17}$$

It is worth pointing out that the standard treatment stops here. If we neglect spinor degrees of freedom, matrix (17) is diagonal in generation space. But, clearly, the diagonalization is not yet complete and we still do not have the propagating degrees of freedom — the normal modes of the theory.

We convert matrix (17) into diagonal form by an extra unitary transformation with the matrix

$$\mathcal{X} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}:$$

$$\mathcal{X} \begin{pmatrix} 0 & \mathcal{M}_d \\ \mathcal{M}_d & 0 \end{pmatrix} \mathcal{X}^\dagger = \begin{pmatrix} \mathcal{M}_d & 0 \\ 0 & -\mathcal{M}_d \end{pmatrix} \tag{18}$$

The complete transformation which diagonalizes H_0^{red} is thus $\mathcal{S} \equiv \mathcal{X} S^{\text{red}}$. After diagonalization, we can write:

$$-\mathcal{L}_{mass} = \frac{1}{2}(\bar{\chi}_+, \bar{\chi}'_-) \begin{pmatrix} \mathcal{M}_d & 0 \\ 0 & -\mathcal{M}_d \end{pmatrix} \begin{pmatrix} \chi_+ \\ \chi'_- \end{pmatrix} \tag{19}$$

or

$$-\mathcal{L}_{mass} = \frac{1}{2}\bar{\chi}_+\mathcal{M}_d\chi_+ - \frac{1}{2}\bar{\chi}'_-\mathcal{M}_d\chi'_- \tag{20}$$

where $\chi_+ \equiv U^T(n_R)^c + U^\dagger n_R$ and $\chi'_- \equiv -U^T(n_R)^c + U^\dagger n_R$. With the redefinition $\chi_- \equiv \gamma_5 \chi'_-$, $-\mathcal{L}_{mass}$ acquires its final diagonal form:

$$-\mathcal{L}_{mass} = \frac{1}{2}\overline{\chi_+}\mathcal{M}_d\chi_+ + \frac{1}{2}\overline{\chi_-}\mathcal{M}_d\chi_- \quad (21)$$

(Note: Rosen [9] also saw the necessity of diagonalizing the 12x12 Hamiltonian and arrived at (20), but suggested that the χ'_- were unphysical eigenstates of negative mass.)

Restoring generation indices, we have:

$$-\mathcal{L}_{mass} = \frac{1}{2}\sum_{k=1}^{2n} m_k \overline{\chi_{+k}}\chi_{+k} + \frac{1}{2}\sum_{k=1}^{2n} m_k \overline{\chi_{-k}}\chi_{-k} \quad (22)$$

where m_k are the diagonal elements of \mathcal{M}_d , and n is the number of generations ($n = 3$). χ_{\pm} are Majorana fields, i.e. $\chi_{\pm}^c = \chi_{\pm}$, so, in the case of n generations we have $4n$ Majorana neutrinos. In the original theory we had $2n$ complex degrees of freedom, while the final theory has $4n$ real degrees of freedom. The particles χ_+ and χ_- have the same masses but may have different symmetry properties and possibly different interactions. We should point out that result (22) differs from the standard results found in much of the literature.

In section (3) we explicitly derived the eigenvectors of the mass matrix in the one generation case. In terms of the weak eigenstates, the propagating eigenvectors of the theory (which are just adjoints of the mass matrix eigenstates (10)) are:

$$\begin{aligned} \chi_{+1} &= -\frac{e^{-i(\varphi-\frac{\phi}{2})}m_+}{M_D}\psi_L + e^{-i\frac{\phi}{2}}(\psi_R)^c - \frac{e^{-i(\frac{\phi}{2}-\varphi)}m_+}{M_D}(\psi_L)^c + e^{i\frac{\phi}{2}}\psi_R \\ \chi_{+2} &= \frac{e^{-i(\varphi-\frac{\phi}{2})}m_-}{M_D}\psi_L + e^{-i\frac{\phi}{2}}(\psi_R)^c + \frac{e^{-i(\frac{\phi}{2}-\varphi)}m_-}{M_D}(\psi_L)^c + e^{i\frac{\phi}{2}}\psi_R \\ \chi'_{-1} &= \frac{e^{-i(\varphi-\frac{\phi}{2})}m_+}{M_D}\psi_L - e^{-i\frac{\phi}{2}}(\psi_R)^c - \frac{e^{-i(\frac{\phi}{2}-\varphi)}m_+}{M_D}(\psi_L)^c + e^{i\frac{\phi}{2}}\psi_R \\ \chi'_{-2} &= -\frac{e^{-i(\varphi-\frac{\phi}{2})}m_-}{M_D}\psi_L - e^{-i\frac{\phi}{2}}(\psi_R)^c + \frac{e^{-i(\frac{\phi}{2}-\varphi)}m_-}{M_D}(\psi_L)^c + e^{i\frac{\phi}{2}}\psi_R \end{aligned} \quad (23)$$

We finally conclude that to obtain the propagating eigenstates for the general case we must diagonalize the 12×12 Hermitian matrix

$$H^{red} = \begin{pmatrix} 0 & 0 & M_L & M_D \\ 0 & 0 & M_D^T & M_R \\ M_L^\dagger & M_D^* & 0 & 0 \\ M_D^\dagger & M_R^\dagger & 0 & 0 \end{pmatrix} \quad (24)$$

A general 12×12 complex Hermitian matrix is diagonalized by a 12×12 unitary matrix. Unitary matrices of this size are parametrized by 66 angles and 78 phases. However, the large numbers of symmetries of the Hamiltonian in this 12 dimensional representation reduce the number of actual angles and phases significantly. After all, M_D , M_R and M_L contain only 18, 12 and 12 real Lagrangian parameters respectively. Algebraic analysis of the Hamiltonian indicates that its characteristic equation is of the form

$$\Pi_{i=1,\dots,6}(\lambda^2 - m_i^2) = 0 \quad (25)$$

with m_i real. m_i can be obtained by a suitable diagonalization of the 6×6 symmetric submatrix

$$\mathcal{M} = \begin{pmatrix} M_L & M_D \\ M_D^T & M_R \end{pmatrix}, \quad (26)$$

but m_i are not the eigenvalues of \mathcal{M} — $\mathcal{M}U = U^*M_d$ and not $\mathcal{M}U = UM_d$.

There are six distinct absolute values of the eigenvalues (we saw that all of them can be made positive). The 6×6 unitary matrices diagonalizing \mathcal{M} are parametrized in terms of 15 angles and 21 phases.

5 Decoupling

Examination of the set of eigenvectors (10) suggests at first sight that the “light” and “heavy” degrees of freedom may not be completely decoupled in the limit of infinitely large M_R . One can see this by inspection of one of the light eigenvectors (an eigenvector which corresponds to a light eigenvalue). The presence of ϕ , which is the phase of a Majorana mass parameter, even after we take limit of large M_R , is the origin of this concern.

In three generations, it is not possible to absorb all 6 phases from M_R by phase rotations on N_{i1} and N_{i2} . Does this imply a non-decoupling theorem? No.

Consider the relevant large M_R limit $M_R/M_D \gg 1$. Let S be the 12×12 matrix whose columns consist of the eigenvectors of M^{red} . S is a unitary matrix, which we can write in the form

$$S = \begin{pmatrix} S_{\ell a} & S_{\ell s} \\ S_{H a} & S_{H s} \end{pmatrix} \quad (27)$$

where the subscripts a and s refer to the “active” (doublet) and “sterile” (singlet) states, and the subscripts ℓ and H refer to the light ($m = \mathcal{O}\left(\frac{M_D^2}{M_R}\right)$) and heavy ($m = \mathcal{O}(M_R)$) states. Since $S_{\ell a}$ and $S_{H s}$ are $\mathcal{O}(1)$, while $S_{\ell s}$ and $S_{H a}$ are $\mathcal{O}\left(\frac{M_D}{M_R}\right)$, the unitarity of S implies that $S_{\ell a}$ is unitary at $\mathcal{O}\left(\frac{M_D^2}{M_R^2}\right)$. Thus, in the limit that $M_R \rightarrow \infty$ when all the light states are massless and degenerate, we can make a unitary transformation (at $\mathcal{O}\left(\frac{M_D^2}{M_R^2}\right)$) to the basis where the active weak eigenstates are the light mass eigenstates, with no remaining phases. This seems to indicate that all physical processes exploring the mixing between active and sterile states *are* suppressed by powers of M_D/M_R .

6 Mixing and CP Violation

The full consequences of a general complex neutrino mass matrix for CP violation in the leptonic sector are currently under investigation. The general mass matrix contains 21 magnitudes and 21 phases (since M_D is a general complex matrix, while M_L and M_R are symmetric). Of these, perhaps 6 phases can be removed by phase redefinitions of the neutrino fields, leaving 15. How many of these are physical? What is the subgroup

of $U(6)$ from which the unitary matrix diagonalizing the 6×6 submatrix \mathcal{M} should be drawn? We reserve most details for future publications.

In particular, let us express the charged current, $J^+ = \overline{\psi_{iL}} \gamma^\mu l_{iL}$ ($\psi_i = \nu_e, \nu_\mu, \nu_\tau$; $l_i = e, \mu, \tau$) in terms of the weak eigenstates:

$$J^+ = \overline{\psi_{iL}} \gamma^\mu l_{iL} = \underbrace{\left(\overline{\psi_{iL}} \overline{(\psi_{iR})^c} \overline{(\psi_{iL})^c} \overline{\psi_{iR}} \right)}_{\overline{\Psi_W^{(\nu)}}} \underbrace{\begin{pmatrix} \gamma^\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_G \otimes \mathbf{I}_3 \underbrace{\begin{pmatrix} l_{iL} \\ (l_{iR})^c \\ (l_{iL})^c \\ l_{iR} \end{pmatrix}}_{\Psi_W^{(l)}} \quad (28)$$

where the subscript “W” denotes a weak eigenstate and \mathbf{I}_3 is three-dimensional unit matrix in generation space. The matrix $\mathcal{S} = \mathcal{X} \mathcal{S}^{\text{red}} = \begin{pmatrix} \mathcal{U}^T & \mathcal{U}^\dagger \\ -\mathcal{U}^T & \mathcal{U}^\dagger \end{pmatrix}$ brings the weak eigenbasis into the mass eigenbasis and we have:

$$J^+ = \overline{\Psi_W^{(\nu)}} \mathcal{S}^{(\nu)\dagger} \mathcal{S}^{(\nu)} G \mathcal{S}^{(l)\dagger} \mathcal{S}^{(l)} \Psi_W^{(l)} = \overline{\Psi_M^{(\nu)}} \mathcal{S}^{(\nu)} G \mathcal{S}^{(l)\dagger} \Psi_M^{(l)}. \quad (29)$$

(We have identified separate transformations matrices $\mathcal{S}^{(\nu)}$ and $\mathcal{S}^{(l)}$ for the neutrinos and for the charged leptons.)

Decompose the matrices $\mathcal{U}^{(\nu)}$ and $\mathcal{U}^{(l)}$ into four block elements each:

$$\mathcal{U}^{(\alpha)} = \begin{pmatrix} \mathcal{U}_{11}^{(\alpha)} & \mathcal{U}_{12}^{(\alpha)} \\ \mathcal{U}_{21}^{(\alpha)} & \mathcal{U}_{22}^{(\alpha)} \end{pmatrix}, \quad (30)$$

where block elements are matrices in generation space (i.e. $\mathcal{U}_{11}^{(\alpha)} \equiv (\mathcal{U}_{11})_{ij} \dots$). Using the fact that $\mathcal{U}^{(l)}$ has block diagonal form ($\mathcal{U}_{12}^{(l)} = \mathcal{U}_{21}^{(l)} = 0$), we can write:

$$J^+ = \begin{pmatrix} \overline{\chi_{+1i}^{(\nu)}} & \overline{\chi_{+2i}^{(\nu)}} & \overline{\chi_{-1i}^{(\nu)'}} & \overline{\chi_{-2i}^{(\nu)'}} \end{pmatrix} \begin{pmatrix} [(\mathcal{U}_{11}^{(\nu)})^T \gamma^\mu (\mathcal{U}_{11}^{(l)})^*]_{ij} & 0 & -[(\mathcal{U}_{11}^{(\nu)})^T \gamma^\mu (\mathcal{U}_{11}^{(l)})^*]_{ij} & 0 \\ [(\mathcal{U}_{12}^{(\nu)})^T \gamma^\mu (\mathcal{U}_{11}^{(l)})^*]_{ij} & 0 & -[(\mathcal{U}_{12}^{(\nu)})^T \gamma^\mu (\mathcal{U}_{11}^{(l)})^*]_{ij} & 0 \\ -[(\mathcal{U}_{11}^{(\nu)})^T \gamma^\mu (\mathcal{U}_{11}^{(l)})^*]_{ij} & 0 & [(\mathcal{U}_{11}^{(\nu)})^T \gamma^\mu (\mathcal{U}_{11}^{(l)})^*]_{ij} & 0 \\ -[(\mathcal{U}_{12}^{(\nu)})^T \gamma^\mu (\mathcal{U}_{11}^{(l)})^*]_{ij} & 0 & [(\mathcal{U}_{12}^{(\nu)})^T \gamma^\mu (\mathcal{U}_{11}^{(l)})^*]_{ij} & 0 \end{pmatrix} \begin{pmatrix} \chi_{+1j}^{(l)} \\ \chi_{+2j}^{(l)} \\ \chi_{-1j}^{(l)'} \\ \chi_{-2j}^{(l)'} \end{pmatrix} \quad (31)$$

According to this, we can write analogs of CKM matrices for neutrinos:

$$\overline{\chi_{+1i}^{(\nu)}} [(\mathcal{U}_{11}^{(\nu)})^T (\mathcal{U}_{11}^{(l)})^*]_{ij} \gamma^\mu (\chi_{+1j}^{(l)} - \chi_{-1j}^{(l)'}) \quad (32)$$

$$\overline{\chi_{+2i}^{(\nu)}} [(\mathcal{U}_{12}^{(\nu)})^T (\mathcal{U}_{11}^{(l)})^*]_{ij} \gamma^\mu (\chi_{+1j}^{(l)} - \chi_{-1j}^{(l)'}) \quad (33)$$

$$- \overline{\chi_{-1i}^{(\nu)'}} [(\mathcal{U}_{11}^{(\nu)})^T (\mathcal{U}_{11}^{(l)})^*]_{ij} \gamma^\mu (\chi_{+1j}^{(l)} - \chi_{-1j}^{(l)'}) \quad (34)$$

$$-\overline{\chi_{-2i}^{(\nu)'}}[(\mathcal{U}_{12}^\nu)^T(\mathcal{U}_{11}^l)^*]_{ij}\gamma^\mu(\chi_{+1j}^{(l)}-\chi_{-1j}^{(l)'}) \quad (35)$$

The essential feature is that all four distinct neutrino mass eigenstates (per generation) are connected to the one combination of the lepton mass eigenstates (which constitutes a lepton of definite handedness). The mixing between the light neutrinos, $\chi_{\pm 1i}$, is given by $[(\mathcal{U}_{11}^\nu)^T(\mathcal{U}_{11}^l)^*]_{ij}$ while the mixing between the heavy neutrinos, $\chi_{\pm 2i}$, is given by $[(\mathcal{U}_{12}^\nu)^T(\mathcal{U}_{11}^l)^*]_{ij}$.

$\mathcal{U}_{11}^\nu \approx 1$ and $\mathcal{U}_{12}^\nu \approx 0$ in the limit of $\frac{M_D}{M_R} \ll 1$. Also, \mathcal{U}_{11}^ν and \mathcal{U}_{12}^ν are unitary only to zeroth order. First order correction spoil their unitarity and so the unitarity of the neutrino CKM matrices. The leptonic analogs of the CKM matrix are thus two 3×3 *nonunitary* matrices in generation space. Even confining oneself to the weak interaction sector of the theory, this implies a much richer structure for CP violation than had hitherto been anticipated. The mixing matrix (and hence the rich CP violating structure) will also appear in other sectors of the theory, in particular the interactions of the charged and neutral leptons with the light Higgs particle(s).

In the pure Dirac case (quark case) $\mathcal{U}_{12}^q = 0$, while \mathcal{U}_{11}^q is unitary, so, we have only one unitary, 3×3 CKM matrix, which is the standard result.

7 Conclusion

With the recent data from the super-Kamiokande collaboration reinforcing so strongly the case for non-zero neutrino masses, the need for a comprehensive generic pedagogical understanding of neutrino masses and mixings is pressing.

We have shown that in the general case of complex Dirac and Majorana mass parameters, the neutrino “mass matrix” (i.e. the rest Hamiltonian) should be thought of as a 24×24 matrix (c.f. equation (4)), although a reduction to 12×12 is immediate. A reduction to a 6×6 mass matrix is possible but one must be cautious in interpreting the results. In particular, the mass eigenvalues are correct, but eigenstates are not. In the usual limit of vanishing Majorana masses (quark case) the standard treatment in terms of a 6×6 matrix can be readily recovered.

We have shown that the mass eigenstates do obey a Majorana condition.

We have argued that the standard see-saw mechanism is preserved even in the case of complex mass matrices so long as the scale of the sterile (right-handed) Majorana masses is large compared to that of the Dirac masses.

We have argued that in the limit $M_D/M_R \ll 1$, the right handed fields decouple from the problem.

Finally, we have briefly hinted at the richness of the problem of mixing and CP violation in the leptonic sector. The group $U(6)$ of 6×6 unitary matrices has 36 parameters. The mass matrix has 21 magnitudes, and 21 phases of which 15 phases are not removable by phase redefinitions. The leptonic CKM matrix is not a 3×3 unitary matrix, so that it can contain more than the usual one CP violating phase.

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Appendix

The Lagrangian density (1) can be readily cast into the more familiar four-component form:

$$-\mathcal{L} = i\bar{\psi}_i \gamma^\mu \partial_\mu \psi_i + \left[\overline{\psi_{iL}} M_{Dij} \psi_{jR} + \frac{1}{2} \overline{(\psi_{iR})^c} M_{Rij} \psi_{jR} + \frac{1}{2} \overline{\psi_{iL}} M_{Lij} (\psi_{jL})^c + \text{h.c.} \right] \quad (36)$$

where ψ is a four-component Dirac spinor,

$$\psi = \begin{pmatrix} \nu_a \\ \bar{N}^{\dot{a}} \end{pmatrix} \quad (37)$$

and we use the Weyl or spinor representation in which

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad (38)$$

In this representation, the projectors onto the left and right subspaces are:

$$P_L = \frac{1}{2} (\mathbf{1} + \gamma_5) \quad \text{and} \quad P_R = \frac{1}{2} (\mathbf{1} - \gamma_5) \quad \text{where} \quad \gamma_5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \quad (39)$$

Thus,

$$\psi_L = P_L \psi = \begin{pmatrix} \nu_a \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_R = P_R \psi = \begin{pmatrix} 0 \\ \bar{N}^{\dot{a}} \end{pmatrix} \quad (40)$$

describe left and right chirality neutrinos. $\bar{\psi}$ is defined as:

$$\bar{\psi} = \psi^\dagger \gamma^0 = (N^a \ \bar{\nu}_{\dot{a}}) \quad , \quad (41)$$

while charge conjugation is defined by the relation:

$$\psi^c = C \bar{\psi}^T \quad \text{where} \quad C = i\gamma^2 \gamma^0 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\bar{\sigma}^2 \end{pmatrix} \quad (42)$$

Thus

$$(\psi_L)^c = \begin{pmatrix} 0 \\ \bar{\nu}^{\dot{a}} \end{pmatrix} \quad \text{and} \quad (\psi_R)^c = \begin{pmatrix} N_a \\ 0 \end{pmatrix} \quad (43)$$

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